A Practical Guide to Market Risk Model Validations - Focusing on VaR and TVaR

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Robust Techniques in Quantitative Finance
University of Oxford, 3-7 September 2018

Outline

It's just a quantile

In mathematical terms, VaR is just a quantile. The quantile function $q_X(\alpha)$ of a random variable X can be defined as follows.

$$q_X(\alpha) = \inf\{x : \mathbb{P}[X \leq x] \geq \alpha\}$$

Using basic equation $\mathbb{P}[X \leq x] = 1 - \mathbb{P}[X > x]$, we can rewrite $q_X(\alpha)$ in the following manner.

$$q_X(\alpha) = \inf\{x : \mathbb{P}[X > x] \leq 1 - \alpha\}$$

In finance, one wants to make sure that the losses (L) do not exceed a certain level with a high probability. This leads to the following definition of $VaR(\alpha)$.

$$VaR(\alpha) = \inf\{l : \mathbb{P}[L > l] \leq 1 - \alpha\}$$

Left and right quantiles symmetry

An alternative definition of VaR based on the P&L process rather than the loss process.

$$VaR(\alpha) = -\sup\{u : \mathbb{P}[P\&L \leq u] \leq 1 - \alpha\}$$

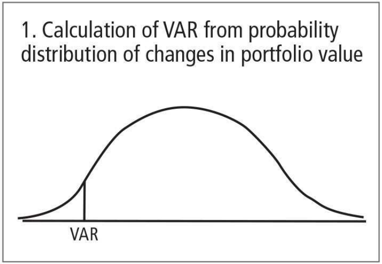
The second definition of VaR is based on the right quantile function.

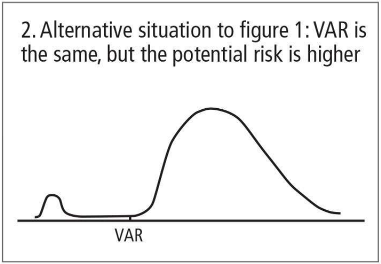
$$q_X^*(\alpha) = \sup\{x : \mathbb{P}[X \leq x] \leq \alpha\}$$

Both VaR definitions agree due to the left and right quantile symmetry.

$$q_X(\alpha) = -q^*_{-X}(1 - \alpha)$$

One VaR - different riskiness





Going further into the tail

VaR gives a lower bound for the extreme unlikely losses. The actual losses can exceed VaR. What's the expected loss in extreme cases? It can be measured as follows.

$$TVaR(\alpha) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR(u)du$$

This can be interpreted as integration over extreme losses with equal weights of $1/(1-\alpha)$. Tail VaR is also known as Expected Shortfall, Conditional VaR, Average VaR, and Expected Tail Loss.

Various VaR applications

- Trading desk limits - $\alpha = .95$, $h = 1$ day, and underlying process is L

$$VaR_{1d}(.95)$$

- Economic capital - $\alpha = .999$, $h = 1$ year, and underlying process is L

$$\text{ECapital} = VaR_{1Y}(.999) - \text{Expected Loss}$$

- Maximum (peak) potential future exposure - $\alpha = .95$, $h = t$ is a variable, and underlying process is V_t^+

$$\text{MPFE} = \max_{0 \le t \le T} VaR_t(.95)$$

Regulatory usage of VaR/TVaR

- 1996-1997 - Basel II (market risk amendment) introduced VaR-based capital concept. It has been also adopted by the FRB in the form of market risk capital rule. VaR-based capital is calculated using $\alpha = .99$, $h = 10$ days, and underlying process L.

$$\text{mRWA} = \max\left\{ VaR_{10d}^{t_0}(.99), \frac{\beta}{60}\sum_{i=1}^{60} VaR_{10d}^{t_0 - i\delta t}(.99) \right\}$$

- 2009-2012 - Basel 2.5 introduced stressed VaR. FRB revised market risk capital rule to include stressed VaR as well.
- 2011-2019 - FRTB introduced TVaR (expected shortfall) to ensure a more prudent capture of "tail risk."

FFIEC 102 report

https://www.ffiec.gov/nicpubweb/nicweb/NicHome.aspx

Last Update: 20180813.101649 RSSD ID: 1073757

Board of Governors of the Federal Reserve System OMB Number 7100-0365
Federal Deposit Insurance Corporation OMB Number 3064-0199
Office of the Comptroller of the Currency OMB Number 1557-0325
Approval expires December 31, 2019
Page 1 of 4

Federal Financial Institutions Examination Council

Market Risk Regulatory Report for Institutions Subject to the Market Risk Capital Rule—FFIEC 102

Report at the close of business June 30, 2018
Month / Day / Year (MRRR 9999)

This report is required by law: 12 U.S.C. § 161 (National banks), 12 U.S.C. § 324 and 12 U.S.C. § 1844(c) (State member banks and Bank holding companies, respectively), 12 U.S.C. § 1467a(b) (Savings and loan holding companies), 12 U.S.C. § 5365 (U.S. Intermediate holding companies); 12 U.S.C. § 1817 (Insured state nonmember commercial and savings banks), and 12 U.S.C. § 1464 (Savings associations).

The FFIEC 102 is to be prepared in accordance with federal regulatory authority instructions. The report must be signed by a senior officer of the reporting entity who can attest that the information submitted in this report meets the requirements set forth in 12 CFR Part 3, Subpart F (OCC); 12 CFR Part 217, Subpart F (Federal Reserve); 12 CFR Part 324, Subpart F (FDIC) (market risk capital rule); and the FFIEC 102 reporting instructions. The senior officer may be the chief financial officer, the chief risk officer, or the equivalent senior officer.

I, the undersigned senior officer of the named bank, bank holding company, savings association, or savings and loan holding company, or U.S. intermediate holding company attest that the FFIEC 102 report for this report date has been prepared in conformance with the instructions issued by the federal regulatory authority and that the reported information meets the requirements set forth in the market risk capital rule to the best of my knowledge and belief.

To fulfill the signature and attestation requirement for the FFIEC 102 for this report date, attach the reporting entity's completed signature page (or a photocopy or a computer-generated version of this page) to the hard-copy record of the data file submitted electronically that the reporting entity must place in its files.

The appearance of the reporting entity's hard-copy record of the submitted data file need not match exactly the appearance of the FFIEC's sample report forms, but should show the caption of each reported item and the reported amount.

Paul Donofrio
Printed Name of Senior Officer (MRRR C490)

Signature of Senior Officer

Chief Financial Officer
Title of Officer (MRRR C491)

08/02/2018
Date of Signature (MM/DD/YYYY) (MRRR J196)

BANK OF AMERICA CORPORATION
Legal Title of Reporting Entity (MRRR 9017)

100 NORTH TRYON STREET
Mailing Address of the Reporting Entity Street / PO Box (MRRR 9110)

CHARLOTTE
City (MRRR 9130)

NC
State Abbreviation (MRRR 9200)

28255
Zip Code (MRRR 9220)

9DJT3UXIJIZJI4WXO774
Legal Entity Identifier (LEI) of the Reporting Entity (Report only if the reporting entity already has an LEI.) (MRRR 9224)

Person to whom questions about this report should be directed:

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12/2016

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FFIEC 102
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Dollar Amounts in Thousands	MRRR	Amount	
Value-at-risk (VaR)-based Capital Requirement			
1. Previous day's VaR-based measure	S298	97,122	1.
2. Average of the immediately preceding 60 business days VaR-based measures	S299	69,906	2.
	MRRR	Number	
3. Multiplication factor: equal to a value of 3.00 or higher (based on backtesting)[1]	S300	3.00	3.
	MRRR	Amount	
4. Greater of item 1 or (item 2 multiplied by item 3)	S301	209,718	4.
Stressed VaR-based Capital Requirement			
5. Most recent stressed VaR-based measure	S302	279,787	5.
6. Item 3 times the average of the preceding 12 weeks stressed VaR-based measures	S303	642,338	6.
7. Greater of item 5 or item 6	S304	642,338	7.
Specific Risk Add-ons			
8. Debt positions	S305	1,310,056	8.
9. Equity positions	S306	368,461	9.
10. For *all* institutions, capital requirements for securitization positions using the Simplified Supervisory Formula Approach (SSFA) or applying a specific risk-weighting factor of 100 percent	S307	817,422	10.
11. For *advanced approaches* institutions, capital requirements for securitization positions using the Supervisory Formula Approach (SFA)	S308	0	11.
12. For *advanced approaches* institutions, capital requirements for securitization positions using the SSFA or applying a specific risk-weighting factor of 100 percent	S309	817,422	12.
13. For *advanced approaches* institutions, sum of items 11 and 12	S310	817,422	13.
14. Standardized measure of specific risk add-ons (sum of items 8, 9, and 10)	S311	2,495,939	14.
15. For *advanced approaches* institutions, advanced measure of specific risk add-ons (sum of items 8, 9, and 13)	S312	2,495,939	15.
Items 16 through 18 are not applicable to an institution that does not calculate a modeled measure of incremental risk.			
Incremental Risk Capital Requirement			
16. Most recent incremental risk measure	S313	365,610	16.
17. Average of the previous 12 weeks measure of incremental risk	S314	400,353	17.
18. Greater of item 16 or item 17	S315	400,353	18.
Items 19 through 51 are not applicable to an institution that does not have a comprehensive risk model; such an institution should go to item 52.			
Comprehensive Risk Capital Requirement			
19. Most recent modeled measure of all price risk	S316	13,708	19.
Standardized Specific Risk Add-ons for *Net Long* Correlation Trading Positions			
20. Debt positions	S319	943,076	20.
21. Equity positions	S320	0	21.
22. For *all* institutions, capital requirements for securitization positions using the SSFA or applying a specific risk-weighting factor of 100 percent	S321	1,071,101	22.
23. For *advanced approaches* institutions, capital requirements for securitization positions using the SFA	S322	353,754	23.
24. For *advanced approaches* institutions, capital requirements for securitization positions using the SSFA or applying a specific risk-weighting factor of 100 percent	S323	0	24.
25. For *advanced approaches* institutions, sum of items 23 and 24	S324	353,754	25.

1. Report the multiplication factor as a number to two decimal places, e.g., 3.00 and 3.75.

03/2016

Coherent risk measures

From the risk measurement perspective, VaR has two major drawbacks - lack of sensitivity to the size of the extreme losses and lack of sub-additivity. A coherent risk measure ρ outperforms VaR in this regard and obeys the following laws.

1. (Positive homogeneity) $\rho(aL) = a\rho(L), \quad \forall a \geq 0$
2. (Sub-additivity) $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$
3. (Normalized) $\rho(L + a1) = \rho(L) + a, \quad \forall a \in \Re$
4. (Monotinicity) $\rho(L_1) \leq \rho(L_2)$ if $L_1 \leq L_2$ in all scenarios

VaR/TVaR Extensions

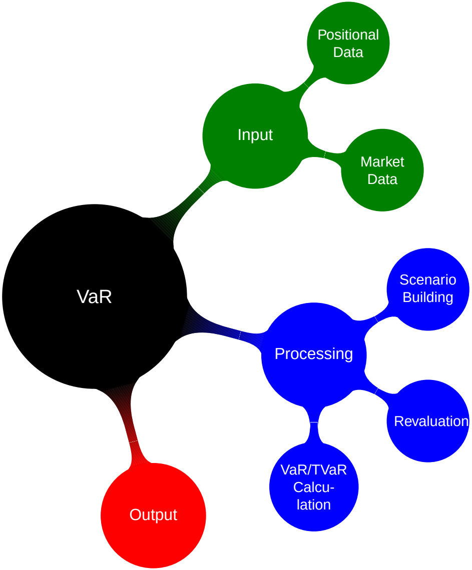
TVaR measure can be generalized by allowing more complicated increasing weighting functions $w(u)$ defined on the interval $[0, 1]$. This leads us to the definition of the spectral risk measure (SRM).

$$SRM(L) = \int_0^1 w(u) VaR_u(L) du$$

The SRM can be re-written using anti-derivative of weighting function $w(u)$ (called distortion function), $D(u) = \int_0^u w(s)ds$. This leads us to the following definition of distortion risk measure (DM).

$$DM(L) = \int_0^1 VaR_u(L) dD(u)$$

VaR/TVaR production process



Common VaR models

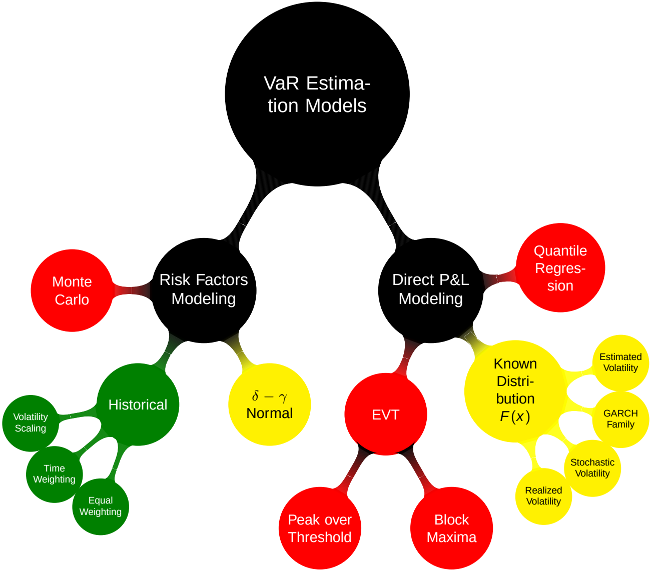
Market-risk practices at 18 financial institutions, 2011, %

Simulation approach	Valuation approach[1]: Sensitivities	Hybrid	Full revaluation
Historical simulation		35	40
Hybrid	5		5
Monte Carlo	15		

1 Banks are deemed to use the sensitivities approach if they use it exclusively, hybrid if they use it at least 30 percent of the time, and full revaluation if less than 30 percent.

Source: McKinsey Market Risk Survey and Benchmarking 2011

VaR models categorization



VaR/TVaR model specifications

Specification attribute	Options
Revaluation methodology	Sensitivity based
	Full revaluation
VaR estimation methodology	Parametric
	Mixed
	Non-parametric
Simulation object	Relative change
	Absolute change
	Log change
Simulation calibration	Historical
	Market implied
Scenario Weighting	Equal
	Exponential
Confidence level scaling	Yes/No
Horizon scaling	Yes/No

What happened on 05/10/2012?

Testimony of Jamie Dimon
Chairman & CEO, JPMorgan Chase & Co.
Before the House Financial Services Committee
Washington, D.C.
June 19, 2012

In December 2011, as part of a firmwide effort in anticipation of new Basel capital requirements, we instructed CIO to reduce risk-weighted assets and associated risk. To achieve this in the synthetic credit portfolio, the CIO could have simply reduced its existing positions; instead, starting in mid-January, it embarked on a complex strategy that entailed adding positions that it believed would offset the existing ones.

JPM VaR vs P&L

JPMORGAN CHASE & CO.
MARKET RISK-RELATED INFORMATION
(in millions)

	1Q12
95% CONFIDENCE LEVEL- AVERAGE IB TRADING VAR, CREDIT PORTFOLIO VAR AND OTHER VAR	
IB VaR by risk type:	
Fixed income	$ 60
Foreign exchange	11
Equities	17
Commodities and other	21
Diversification benefit to IB trading VaR (a)	(46)
IB trading VaR (b)	63
Credit portfolio VaR (c)	32
Diversification benefit to IB trading and credit portfolio VaR (a)	(14)
Total IB trading and credit portfolio VaR	81
Other VaR:	
Mortgage Production and Servicing VaR (d)	11
Chief Investment Office VaR (e)(f)	129
Diversification benefit to other VaR (a)	(4)
Total other VaR	136
Diversification benefit to total IB and other VaR (a)	(47)
Total IB and other VaR (f)(g)	$ 170

JPMORGAN CHASE & CO.
CORPORATE/PRIVATE EQUITY
FINANCIAL HIGHLIGHTS
(in millions, except headcount data)

	2Q12
INCOME STATEMENT	
REVENUE	
Principal transactions	$ (3,576)
Securities gains	1,013
All other income	159
Noninterest revenue	(2,404)
Net interest income	(205)
TOTAL NET REVENUE (a)	(2,609)
Provision for credit losses	(11)
NONINTEREST EXPENSE	
Compensation expense	652
Noncompensation expense (b)	1,317
Subtotal	1,969
Net expense allocated to other businesses	(1,410)
TOTAL NONINTEREST EXPENSE	559
Income/(loss) before income tax expense/(benefit)	(3,157)
Income tax expense/(benefit)	(1,380)
NET INCOME/(LOSS)	$ (1,777)
MEMO:	
REVENUE	
Private Equity	$ 410
Treasury and Chief Investment Office ("CIO")	(3,434)
Other Corporate	415
TOTAL NET REVENUE	$ (2,609)
NET INCOME/(LOSS)	
Private Equity	$ 197
Treasury and CIO	(2,078)
Other Corporate	104
TOTAL NET INCOME/(LOSS)	$ (1,777)
TOTAL ASSETS (period-end)	$ 667,206
Headcount	23,020

CIO strategy

- JPMorgan's CIO sold substantial amounts of CDX IG index exposure in the first quarter of the year. It also bought CDX HY protection, with total notional trade sizes running to tens of billions of dollars.
- In the economic downturn scenario, HY companies' spreads will widen more than those of IG companies.
- A substantial divergence occurred between IG and HY indices. The two indices moved in tandem in the past. The historical relationship between the indices has been broken.

CDX IG and CDX HY correlation breakdown

In periods of heightened market volatility, risk factors' correlations can differ substantially from those seen in normal periods. This is so-called "correlation breakdown" effect.

Sample VaR - 200-year-old problem

A number of banks utilize historical approach. A common choice for the VaR estimator is the sample quantile.

$$V_1 := F_n^{-1}(p) = X_{(np)}$$

In the last part of the Second Supplement (1818) to the monumental Théorie Analytique des Probabilités, Laplace derived the asymptotic distribution of a single order statistic. Laplace then compared the sample mean and median estimators on the basis of the variances of their asymptotic distributions. It is now well known that $X_{(np)}$ is asymptotically normally distributed.

$$\sqrt{n}(V_1 - F^{-1}(p)) \overset{dist}{\to} N(0, p(1-p)/f^2(F^{-1}(p))), \qquad n \to \infty$$

Sample TVaR

Now turning towards the nonparametric estimator of TVaR, the sample tail T_1.

$$T_1 \; := \; \frac{1}{n(1-p)} \sum_{i=(np)+1}^{n} X_{(i)}$$

It can be shown that.

$$\sqrt{n}(T_1 - TVaR(p)) \overset{dist}{\rightarrow} N\left(0, \tau_1^2\right)$$

$$\tau_1^2 \; := \; \frac{1}{(1-p)^2} \int_{F^{-1}(p)}^{\infty} (u - VaR(p))^2 dF(u) - (TVaR(p) - VaR(p))^2$$

Theoretical VaR/TVaR under normal and Student's t distributions

If the loss L has normal $N(\mu, \sigma^2)$ or Student's t ($t(\nu, \mu, \sigma^2)$) distributions, the VaR and TVaR can be expressed in the following manner.

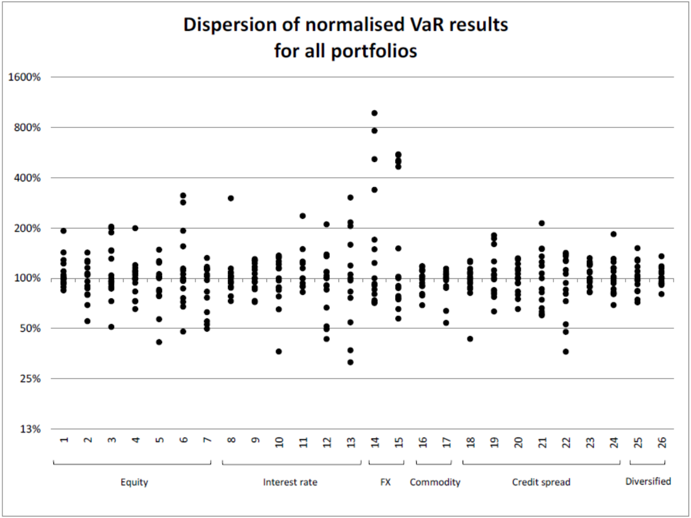
$$VaR_N(\alpha) = \mu + \sigma\Phi^{-1}(\alpha)$$

$$VaR_t(\alpha) = \mu + \sigma t_\nu^{-1}(\alpha)$$

$$TVaR_N(\alpha) = \mu + \frac{\sigma}{1-\alpha}\varphi(\Phi^{-1}(\alpha))$$

$$TVaR_t(\alpha) = \mu + \frac{\sigma}{1-\alpha} g_\nu(t_\nu^{-1}(\alpha)) \left(\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1} \right)$$

VaR model variance



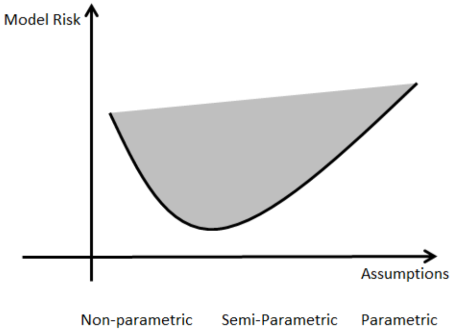
Statistical properties of VaR/TVaR estimators, elicitability

Financial industry tends to assess the quality of the VaR models through back testing. However, back testing is challenging when dealing with TVaR. It has been shown that VaR is elicitable whereas TVaR is not.

Model validation teams need to address the theoretical soundness of the VaR/TVaR estimation models. They need to answer the following questions. How good are the standard non-parametric estimators, as well as their semi-parametric and parametric counterparts? Are they unbiased, sufficient, consistent, asymptotically efficient? What are the "best" estimators of VaR/TVaR?

In our analysis we will focus on the asymptotic efficiency of the VaR and TVaR estimators. This will allow us to avoid the pitfalls of the back testing approach. We will then put proposed VaR estimators to the real test.

VaR model risk smile



Testing framework

	VaR(Model,p)	TVaR(Model,p)
Theo		
P	$V_{1,2,3}$	$V_{1,2,3}$
SP	$R_{1,2}$	$R_{1,2}$
NP	MME/MLE/UMVU	MME/MLE/UMVU

Enhanced non-parametric VaR/TVaR estimators

Theorem. (Joint asymptotic distribution of L-estimators)

Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} F$ with the corresponding order statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. For piecewise smooth functions $L_1(x), L_2(x)$, $x \in [0,1]$ consider the order statistics $\widehat{\Theta}_{1n} = \frac{1}{n}\sum_{i=1}^{n} L_1(\frac{i}{n+1})X_{(i)}$, $\widehat{\Theta}_{2n} = \frac{1}{n}\sum_{i=1}^{n} L_2(\frac{i}{n+1})X_{(i)}$. If F is absolutely continuous with respect to Lebesgue measure with $E(X_1^2) < \infty$ then we can get.

$$\sqrt{n}\begin{pmatrix} (X_{(np)} - F^{-1}(p)) \\ \widehat{\Theta}_{1n} - J_1(L_1, F) \\ \widehat{\Theta}_{2n} - J_2(L_2, F) \end{pmatrix} \overset{dist}{\rightarrow} N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma\right)$$

Robust non-parametric VaR estimators - 2nd variation

An improvement in efficiency may be possible under the location-scale model, $F(x) = G((x-\mu)/\sigma)$, when G has a symmetric density. Optimizing $M_n + \alpha X_{(np)} - (1-\alpha)X_{(n(1-p))}$ we get.

$$V_2 := M_n + \frac{X_{(np)} - X_{(n(1-p))}}{2}$$

where M_n is the sample median. It is also a consistent and asymptotically normal estimator with the following asymptotic variance.

$$\sigma_2^2 \ = \ \sigma^2 \left\{ \frac{1}{4(g(0))^2} + \frac{(1-p)(2p-1)}{2(g(G^{-1}(p)))^2} \right\}$$

Robust non-parametric VaR estimators - 3rd variation

By expanding on the media portion, we obtain another consistent and asymptotically normal estimator.

$$V_3 \; := \; \beta X_{(\gamma n)} + (1-2\beta) X_{(0.5n)} + \beta X_{(n(1-\gamma))} + \frac{X_{(np)} - X_{(n(1-p))}}{2},$$

where $\beta, \gamma \leq \frac{1}{2}$.

$$\sigma_3^2 \; = \; \sigma^2 \left\{ \frac{2\beta^2\gamma}{(g(G^{-1}(\gamma)))^2} + \frac{(1-2\beta)^2}{4\,(g(0))^2} + \frac{2\beta(1-2\beta)\gamma}{g(0)\,g(G^{-1}(\gamma))} + \frac{(1-p)(2p-1)}{2\,(g(G^{-1}(p)))^2} \right\}$$

Further refinements could be achieved by more sophisticated estimators of order statistics, however, the resulting improvements may be minor and parametric model dependent, if any.

Robust non-parametrics TVaR estimators - 2nd and 3rd variations

Just as in the case of $VaR(p)$, one may modify sample TVaR estimator a bit further when the distribution G has a symmetric density, g.

$$T_2 \ := \ M_n + \frac{1}{2n(1-p)} \left(\sum_{i=(np)+1}^{n} X_{(i)} - \sum_{i=1}^{(n(1-p))} X_{(i)} \right)$$

$$\begin{aligned} T_3 \ := \ & \beta X_{(\gamma n)} + (1-2\beta) X_{(0.5n)} + \beta X_{(n(1-\gamma))} \\ & + \frac{1}{2n(1-p)} \left(\sum_{i=(np)+1}^{n} X_{(i)} - \sum_{i=1}^{(n(1-p))} X_{(i)} \right), \end{aligned}$$

for $\beta, \gamma \leq \frac{1}{2}$. These estimators are consistent and asymptotically normal with the following asymptotic variances.

Robust non-parametrics TVaR estimators - 2nd and 3rd variations

$$\tau_3^2 = \sigma^2 \left\{ \frac{2\beta^2\gamma}{(g(G^{-1}(\gamma)))^2} + \frac{(1-2\beta)^2}{4\,(g(0))^2} + \frac{2\beta(1-2\beta)\gamma}{g(0)\,g(G^{-1}(\gamma))} \right\} + \frac{1}{2}\left\{ \frac{1}{(1-p)^2} \int_{F^{-1}(p)}^{\infty} (u - VaR(p))^2 dF(u) \right\} - (TVaR_p - VaR(p))^2$$

For $\beta = 0$ we get the asymptotic variance τ_2^2 for the estimator T_2.

Following this idea of separately estimating the location and scale terms, one may build more sophisticated estimators, however, the gains may or may not be worth the effort, and the optimal choices may be model dependent.

Parametric VaR estimators

The estimators presented earlier are non-parametric ones. They do not require any assumptions about the underlying distributions. One can also use parametric approach to derive VaR and TVaR. Let's consider standard examples of normal distribution and t-distribution with unknown parameters. The quantiles can be expressed as follows.

$$VaR_N^{UMVU}(\alpha) = \overline{X}_n + \frac{S\sqrt{n-1}}{D_n}\Phi^{-1}(\alpha)$$

where $D_n = \mathbb{E}\left[\sqrt{\chi^2_{(n-1)}}\right]$.

$$VaR_t^{UMVU}(\alpha) = \overline{X}_n + \frac{S\sqrt{n-1}}{D_n(1-\alpha)}\varphi\left(\Phi^{-1}(\alpha)\right)$$

Semi-parametric VaR/TVaR estimators

Now consider a location-scale representation of F in terms of random variables, $X_i = \mu + \sigma U_i$, where $U_1, U_2, \cdots, U_n \overset{iid}{\sim} G$, and $E(U_i) = 0$, with $Var(U_i) = \gamma_G^2 =: \gamma^2$. We will make the blanket assumption that $E(X_1^4) < \infty$. However, some of the results hold with only assuming finite variance. Since both $VaR(p)$ and $TVaR(p)$ are of the form, $\mu + \sigma c(p)$, with $E(U_1^2) > 0$, for appropriate expressions for $c(p)$ that depend on G, we consider estimators of $\mu + \sigma c(p)$.

Semi-parametric VaR/TVaR estimators - 1st variation

Applying the method of moments estimation technique to the empirical distribution function F_n we may consider a very simple method of moments semi-parametric estimator (MME)

$$R_1 \;=\; \overline{X}_n + \frac{c(p)}{\gamma} S$$

When $c(p)$ is available then one gets asymptotic normality and not just consistency.

$$\sqrt{n}(R_1 - (\mu + c(p)\sigma)) \overset{dist}{\to} N(0, \nu_1^2)$$

$$\nu_1^2 := \sigma^2 \left\{ \gamma^2 + 0.25(c(p))^2(\kappa_G - 1) + \gamma c(p)\psi_G \right\},$$

where ψ_G and κ_G are the skewness and kurtosis of the distribution G.

Semi-parametric VaR/TVaR estimators - 2nd variation

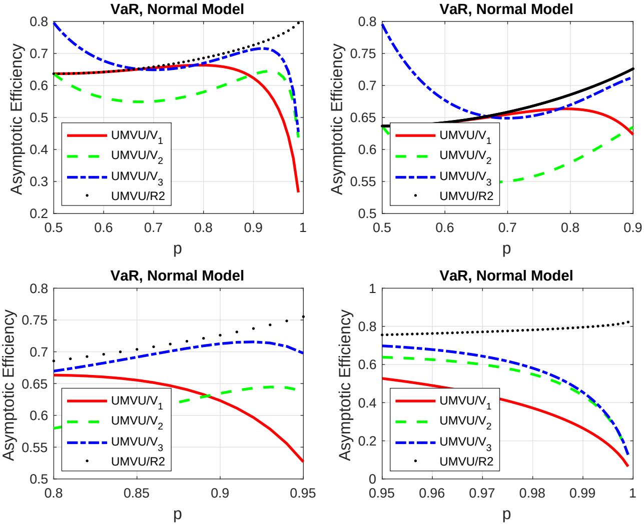
One may consider the following modification of the semi-parametric estimator for $VaR(p)$ or $TVaR(p)$.

$$R_2 \ := \ M_n + \frac{Y_n}{\delta} c(p),$$

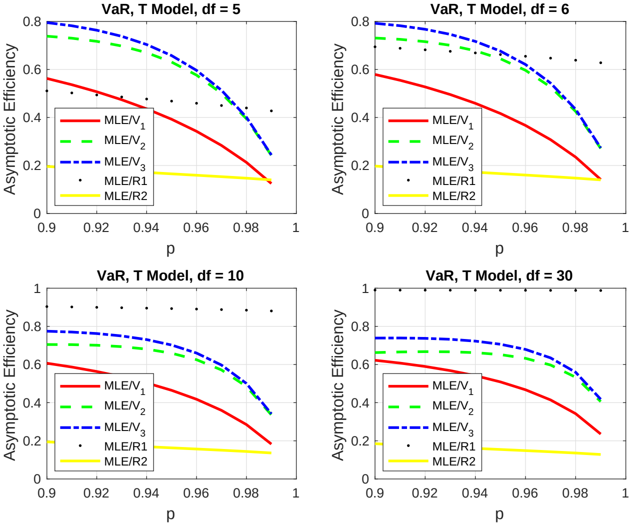
where $\delta = E|U_1 - M_G|$, M_G is the median and $Y_n = \frac{1}{n}\sum_{i=1}^{n}|X_i - M_n|$. When $c(p)$ is available then one gets asymptotic normality.

$$\nu_2^2 := \sigma^2 \left\{ \frac{1}{4(g(M_G))^2} + \frac{Var(|U_1 - M_G|)(c(p))^2}{\delta^2} - \frac{c(p)\,M_G}{\delta g(M_G)} \right\}$$

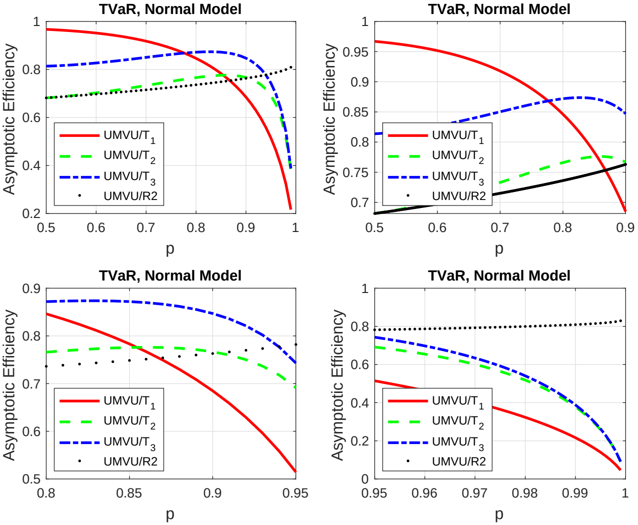
Comparative analysis - VaR for normal model



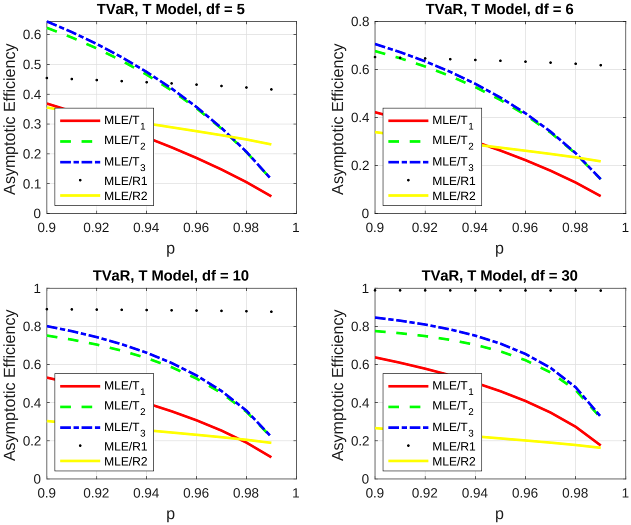
Comparative analysis - VaR for Student’s t model



Comparative analysis - TVaR for normal model



Comparative analysis - TVaR for Student's t model



Sample variances of VaR estimators (2009-2018)

Tickers	V_1	V_2	V_3	R_2	UMVU
DB Commodity	58.54	43.98	45.48	38.65	42.64
Emerging Markets	205.68	171.62	175.81	112.79	211.33
High Yield	38.31	26.50	26.59	17.60	40.95
Developed (exc. North America)	97.72	80.72	81.07	63.39	98.32
Short Term Treasury	0.04	0.03	0.03	0.03	0.07
S&P 500	69.03	60.19	62.58	43.80	75.62

VaR breaches (2018)

Tickers	V_1	V_2	V_3	R_2	UMVU
DB Commodity	8.0	7.0	7.0	10.0	8.0
Emerging Markets	15.0	15.0	15.0	18.0	15.0
High Yield	6.0	6.0	6.0	8.0	5.0
Developed (exc. North America)	11.0	12.0	12.0	13.0	11.0
Short Term Treasury	7.0	7.0	7.0	7.0	7.0
S&P 500	16.0	16.0	16.0	16.0	16.0

Conclusions

- Simple nonparametric estimators of $VaR(p)$ and $TVaR(p)$, V_1, T_1 can have extremely low efficiency
- One can improve stability of the simple nonparametric estimators by using more order statistics
- Semi parametric estimators of $VaR(p)$ and $TVaR(p)$ can give some protection against the model risk and still have reasonably high efficiency
- While taking a considerable model risk, uniformly minimum variance unbiased or maximum likelihood estimators of $VaR(p)$ and $TVaR(p)$ can be constructed under some typical parametric models
- By using alternative estimators, one can both improve the back testing results and minimize the variance. This would lead to a more accurate and stable capital calculations